

THE INTERPLAY OF ALGEBRA AND GEOMETRY IN THE SETTING OF REGULAR ALGEBRAS

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ABSTRACT. This article is based on a talk given by the author at MSRI in the workshop *Connections for Women* in January 2013, while being a part of the program *Noncommutative Algebraic Geometry and Representation Theory* at MSRI. One purpose of the exposition is to motivate and describe the geometric techniques introduced by M. Artin, J. Tate and M. Van den Bergh in the 1980s at a level accessible to graduate students. Additionally, some advances in the subject since the early 1990s are discussed, including a recent generalization of complete intersection to the noncommutative setting, and the notion of graded skew Clifford algebra and its application to classifying quadratic regular algebras of global dimension at most three. The article concludes by listing some open problems.

INTRODUCTION

Many non-commutative algebraists in the 1980s were aware of the successful marriage of algebra and algebraic geometry in the commutative setting and wished to duplicate that relationship in the non-commutative setting. One such line of study was the search for a subclass of non-commutative algebras that “behave” enough like polynomial rings that a geometric theory could be developed for them. One proposal for such a class of algebras are the *regular* algebras, introduced in [2], that were investigated using new geometric techniques in the pivotal papers of M. Artin, J. Tate and M. Van den Bergh ([3, 4]).

About the same time, advances in quantum mechanics in the 20th century had produced many new non-commutative algebras on which traditional techniques had only yielded limited success, so a need had arisen to find new techniques to study such algebras (c.f., [13, 16, 31, 32, 43]). One such algebra was the Sklyanin algebra, which had emerged from the study

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of quantum statistical mechanics ([31, 32]). By the early 1990s, T. Levasseur, S. P. Smith, J. T. Stafford and others had solved the 10-year old open problem of completely classifying all the finite-dimensional irreducible representations (simple modules) over the Sklyanin algebra, and their methods were the geometric techniques developed by Artin, Tate and Van den Bergh ([22, 33, 34]).

Concurrent with the above developments, another approach was considered via differential geometry and deformation theory to study the algebras produced by quantum physics. That approach is the study of certain non-commutative algebras via Poisson geometry (c.f., [12]). At the heart of both approaches are homological and categorical techniques, so it is perhaps no surprise that the two approaches have much overlap; often, certain geometric objects from one approach are in one-to-one correspondence with various geometric objects from the other approach (depending on the algebra being studied – c.f., [44, 45, 46]). A survey of recent advances in Poisson geometry may be found in [15].

Given the above developments, the early 1990s welcomed a new era in the field of non-commutative algebra in which geometric techniques took center stage. Since that time, the subject has spawned many new ideas and directions, as demonstrated by the MSRI programs in 2000 and 2013.

This article is based on a talk given by the author in the *Connections for Women* workshop held at MSRI in January 2013 and it has two objectives. The first is to motivate and describe the geometric techniques of Artin, Tate and Van den Bergh at a level accessible to graduate students, and the second is to discuss some developments towards the attempted classification of quadratic regular algebras of global dimension four, while listing open problems. An outline of the article is as follows.

Section 1 concerns the motivation and development of the subject, with emphasis on quadratic regular algebras of global dimension four. Section 2 discusses constructions of certain types of quadratic regular algebras of arbitrary finite global dimension, with focus on graded Clifford algebras and graded skew Clifford algebras. This section also discusses a new type of symmetry for square matrices called μ -symmetry. We conclude this section by revisiting the classification of quadratic regular algebras of global dimension at most three, since almost all such algebras may be formed from regular graded skew Clifford algebras. In Section 3, we discuss geometric techniques that apply to graded Clifford algebras and graded skew Clifford algebras in order to determine when those algebras are regular. This section also considers the issue of complete intersection in the non-commutative setting. We conclude with Section 4 which lists some open problems and related topics.

Although the main objects of study from [3, 4] are discussed in this article, several topics from [3, 4] are omitted; for surveys of those topics, the reader is referred to [35, 36] and to D. Rogalski’s lecture notes, [26], from the graduate workshop “Noncommutative Algebraic Geometry” at MSRI in June 2012.

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1. THE GEOMETRIC OBJECTS

In this section, we discuss the motivation and development of the subject, with emphasis on quadratic regular algebras of global dimension at most four.

Throughout this section, \mathbb{k} denotes an algebraically closed field and, for any graded algebra B , the span of the homogeneous elements of degree i will be denoted by B_i .

1.1. Motivation.

Consider the \mathbb{k} -algebra, S , on generators z_1, \dots, z_n with defining relations:

$$z_j z_i = \mu_{ij} z_i z_j, \quad \text{for all distinct } i, j,$$

where $0 \neq \mu_{ij} \in \mathbb{k}$ for all i, j , and $\mu_{ij}\mu_{ji} = 1$ for all distinct i, j . If $\mu_{ij} = 1$, for all i, j , then S is the commutative polynomial ring and has a rich subject of algebraic geometry associated with it; in particular, by the (projective) Nullstellensatz, the points of $\mathbb{P}(S_1^*)$ are in one-to-one correspondence with certain ideals of S via $(\alpha_1, \dots, \alpha_n) \leftrightarrow \langle \alpha_i z_1 - \alpha_1 z_i, \dots, \alpha_i z_n - \alpha_n z_i \rangle$, where $\alpha_i \neq 0$. Before continuing, we first observe that for such an ideal I , the graded module S/I has the property that its Hilbert series is $H(t) = 1/(1-t)$ and that S/I is a 1-critical (with respect to GK-dimension) graded cyclic module over S .

However, if $\mu_{ij} \neq 1$ for any i, j , then S still “feels” close to commutative, and one would expect there to be a way to relate algebraic geometry to it. The geometric objects in [3] are modelled on the module S/I above; instead of using actual points or lines etc, certain graded modules are used as follows.

1.2. Points, Lines, etc.

Definition 1.1. [3] Let $A = \bigoplus_{i=0}^{\infty} A_i$ denote an \mathbb{N} -graded, connected (meaning $A_0 = \mathbb{k}$) \mathbb{k} -algebra generated by A_1 where $\dim(A_1) = n < \infty$. A graded right A -module $M = \bigoplus_{i=0}^{\infty} M_i$ is called a *right point module* (respectively, *line module*) if:

- (a) M is cyclic with $M = M_0 A$, and
- (b) $\dim_{\mathbb{k}}(M_i) = 1$ for all i (respectively, $\dim_{\mathbb{k}}(M_i) = i + 1$ for all i).

If A is the polynomial ring S , then the module S/I from §1.1 is a point module. In general, one may associate some geometry to point and line modules as follows. Condition (a) implies that A maps onto M via $a \mapsto ma$, for all $a \in A$, where $\{m\}$ is a \mathbb{k} -basis for M_0 , and this map restricts via the grading to a linear map $\theta : A_1 \rightarrow M_1$. Let $K \subset A_1$ denote the kernel of θ . Condition (b) implies that $\dim_{\mathbb{k}}(K) = n - 1$ (respectively, $n - 2$), so that $K^\perp \subset A_1^*$ has dimension one (respectively, two). Thus, $\mathbb{P}(K^\perp)$ is a point (respectively, a line) in the geometric space $\mathbb{P}(A_1^*)$.

The Hilbert series of a point module is $H(t) = 1/(1 - t)$, whereas the Hilbert series of a line module is $1/(1 - t)^2$. Hence, a plane module is defined as in Definition 1.1 but condition (b) is replaced by the requirement that the module have Hilbert series $1/(1 - t)^3$ (c.f., [3]). Similarly, one may define d -linear modules, where the definition is modelled on Definition 1.1, but the module has Hilbert series $1/(1 - t)^{d+1}$ (c.f., [29]).

For many algebras, d -linear modules are $(d + 1)$ -critical with respect to GK-dimension. This leads to the following generalization of a point module.

Definition 1.2. [10] With A as in Definition 1.1, we define a *right base-point module* over A to be a graded 1-critical (with respect to GK-dimension) right A -module M such that $M = \bigoplus_{i=0}^{\infty} M_i = M_0 A$ and M has Hilbert series $H_M(t) = c/(1 - t)$ for some $c \in \mathbb{N}$.

If $c = 1$ in Definition 1.2, then the module is a point module; whereas if $c \geq 2$, then the module is called a fat point module ([1]). The only base-point modules over the polynomial ring are point modules. On the other hand, in general, the algebra S from §1.1 can have fat point modules, so fat point modules are viewed as generalizations of points, and this is made more precise in [1].

In [3], Artin, Tate and Van den Bergh proved that, under certain conditions, the point modules are parametrized by a scheme; that is, there is a scheme that represents the functor of point modules. Later, in [48], this scheme was called the *point scheme*. A decade later, in [29], it was proved by B. Shelton and the author that (under certain conditions) d -linear modules are parametrized by a scheme; that is, there is a scheme that represents the functor of d -linear modules. If $d = 0$, then this scheme is isomorphic to the point scheme; if $d = 1$, the scheme is called the *line scheme*.

By factoring out a nonzero graded submodule from a point module, one obtains a truncated point module as follows.

Definition 1.3. [3] With A as in Definition 1.1, we define a *truncated right point module of length m* to be a graded right A -module $M = \bigoplus_{i=0}^{m-1} M_i$ such that M is cyclic, $M = M_0 A$ and $\dim_{\mathbb{k}}(M_i) = 1$ for all $i = 0, \dots, m - 1$.

For many quadratic algebras A , there exists a one-to-one correspondence between the truncated point modules over A of length three and the point modules over A . Moreover, if the

algebra A in Definition 1.3 is quadratic, then the truncated point modules of length three are in one-to-one correspondence with the zero locus in $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ of the defining relations of A . To see this, we fix a \mathbb{k} -basis $\{x_1, \dots, x_n\}$ for A_1 , and use T to denote the free \mathbb{k} -algebra on x_1, \dots, x_n , and let $Z \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ denote the zero locus of the defining relations of A . Viewing each x_i as the i 'th coordinate function on A_1^* , let $p = (\alpha_i) \in \mathbb{P}(A_1^*)$ and $r = (\beta_i) \in \mathbb{P}(A_1^*)$, where $\alpha_i, \beta_i \in \mathbb{k}$ for all $i = 1, \dots, n$. Let $M = \mathbb{k}v_0 \oplus \mathbb{k}v_1 \oplus \mathbb{k}v_2$ denote a three-dimensional vector space that is a T -module via the action determined by

$$v_0x_i = \alpha_i v_1, \quad v_1x_i = \beta_i v_2, \quad v_2x_i = 0,$$

for all i . It follows that M is a truncated point module over T of length three. If $g \in T_2$, then $v_1g = 0 = v_2g$ and $v_0g = g(p, r)v_2$. In particular, if $f \in T_2$ is a defining relation of A , then $Mf = 0$ if and only if $f(p, r) = 0$. Hence, M is an A -module if and only if $(p, r) \in Z$. This one-to-one correspondence between Z and truncated point modules of length three also exists at the level of schemes; the reason being that the scheme Z represents the functor of truncated point modules of length three. The method of proof of this is to repeat the preceding argument for a truncated point module of length three over $R \otimes_{\mathbb{k}} T$ and $R \otimes_{\mathbb{k}} A$, where R is a commutative \mathbb{k} -algebra, together with localization techniques; for details the reader is referred to [3, Proposition 3.9], its proof, and the paragraph preceding that result. This correspondence will be revisited in §1.4.

For completeness, we finish this subsection with some technical definitions that play minor roles throughout the text. The reader is referred to [21, 22] for details and for results concerning algebras satisfying these definitions.

Definition 1.4. [21, Definition 2.1] A noetherian ring B is called *Auslander-regular* (respectively, *Auslander-Gorenstein*) if

- (a) the global homological dimension (respectively, (left and right) injective dimension) of B is finite, and
- (b) every finitely generated B -module M satisfies the *Auslander condition*, namely, for every $i \geq 0$ and for every B -submodule N of $\text{Ext}_B^i(M, B)$, we have $j(N) \geq i$, where $j(N) = \inf\{\ell : \text{Ext}_B^\ell(N, B) \neq 0\}$.

Definition 1.5. [21, Definition 5.8] A noetherian \mathbb{k} -algebra B of integral GK-dimension n satisfies the *Cohen-Macaulay property* if $\text{GKdim}(M) + j(M) = n$ for all nonzero finitely generated B -modules M .

1.3. Regular Algebras.

The goal of [3] was to classify, in a user-friendly way, the generic regular algebras of global dimension three that were first analysed in [2]. In [3], such algebras were shown to be noetherian by using the geometric techniques developed in [3]. Regular algebras are often viewed as non-commutative analogues of polynomial rings and are defined as follows.

Definition 1.6. [2] A finitely generated, \mathbb{N} -graded, connected \mathbb{k} -algebra $A = \bigoplus_{i=0}^{\infty} A_i$, generated by A_1 , is *regular* (or *AS-regular*) of global dimension r if

- (a) it has global homological dimension $r < \infty$, and
- (b) it has polynomial growth (i.e., there exist positive real numbers c and δ such that $\dim_{\mathbb{k}}(A_i) \leq ci^{\delta}$ for all i), and
- (c) it satisfies the *Gorenstein condition*, namely, a minimal projective resolution of the left trivial module ${}_A\mathbb{k}$ consists of finitely generated modules and dualizing this resolution yields a minimal projective resolution of the right trivial module $\mathbb{k}_A[e]$, shifted by some degree e .

Although all three conditions in Definition 1.6 are satisfied by the polynomial ring, the main reason a regular algebra is viewed as a non-commutative analogue of a polynomial ring is due to condition (c), since it imposes a symmetry condition on the algebra that replaces the symmetry condition of commutativity. The reader should note that, in the literature, (c) is sometimes replaced by an equivalent condition that makes the symmetry property less obvious; namely, $\text{Ext}_A^i({}_A\mathbb{k}, A) \cong \delta_r^i \mathbb{k}_A[e]$, where δ_r^i is the Krönercker-delta symbol. An \mathbb{N} -graded connected \mathbb{k} -algebra that is generated by degree-1 elements and which is Auslander-regular with polynomial growth is AS-regular ([21]). For a notion of regular algebra where the algebra is not generated by degree-1 elements, see [6, 7, 37, 38, 39, 40].

Examples 1.7.

- (a) The algebra S from §1.1 is regular.
- (b) If $\mathbb{k} = \mathbb{C}$, then many algebras from physics are regular. In particular, homogenizations of universal enveloping algebras of finite-dimensional Lie algebras, the coordinate ring of quantum affine n -space, the coordinate ring of quantum $m \times n$ matrices, and the coordinate ring of quantum symplectic n -space are all regular ([19, 20, 23])
- (c) If the global dimension of a regular algebra is one, then the algebra is the polynomial ring on one variable. However, by [2], if the global dimension is two, then there are two types of such algebra as follows. For both types, the algebra has two generators, x, y , of degree one and one defining relation f , where either $f = xy - yx - x^2$ (*Jordan plane*) or $f = xy - qyx$ (*quantum affine plane*), where $q \in \mathbb{k}$ can be any nonzero scalar.

However, if the global dimension is three, then the situation is much richer; some of the algebras are quadratic with three generators and three defining relations, whereas the rest

have two generators and two cubic relations ([2]). Such algebras that are generic are classified in [3] according to their point schemes, and in all cases, the point scheme is the graph of an automorphism σ . Moreover, the algebra is a finite module over its center if and only if σ has finite order.

1.4. Global Dimension Four.

Although many regular algebras of global dimension four have been extensively studied, there is no classification yet. Recently, the progress towards classifying non-quadratic regular algebras of global dimension four made good headway via the work in [24, 27]. However, quadratic regular algebras of global dimension four constitute most of the regular algebras of global dimension four, so their attempted classification is one of the motivating problems that drives the subject forward. We end this section by summarizing some key results for this latter case; in this setting, the algebra has four generators and six relations.

In unpublished work, Van den Bergh proved in the mid-1990s that any quadratic (not necessarily regular) algebra A on four generators with six generic defining relations has twenty (counted with multiplicity) nonisomorphic truncated point modules of length three. Hence, A has at most twenty nonisomorphic point modules. He also proved that if, additionally, A is Auslander-regular of global dimension four, then A has a 1-parameter family of line modules. For lack of a suitable reference, we outline the proof of these results. Let $M(4, \mathbb{k})$ denote the space of 4×4 matrices with entries in \mathbb{k} . For the first result, we write points of $\mathbb{P}(A_1^*)$ as columns and, by mapping $(a, b) \in \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ to the matrix $ab^T \in M(4, \mathbb{k})$, we have that $\mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ is isomorphic to the scheme Ω_1 of rank-1 elements in $\mathbb{P}(M(4, \mathbb{k}))$. Correspondingly, the defining relations of A map to homogeneous degree-1 polynomial functions on $M(4, \mathbb{k})$, and their zero locus $Z' \subset \mathbb{P}(M(4, \mathbb{k}))$ can be identified with a \mathbb{P}^9 . With these identifications, the zero locus $Z \subset \mathbb{P}(A_1^*) \times \mathbb{P}(A_1^*)$ of the defining relations of A is isomorphic to $\Omega_1 \cap Z' \subset \mathbb{P}(M(4, \mathbb{k}))$. Since Ω_1 has dimension six and degree twenty, $\dim(Z) \geq 6 + 9 - 15 = 0$, and, by Bézout's Theorem, $\deg(Z) = 20$. Hence, generically, Z is finite with twenty points, so the first result follows by using the discussion after Definition 1.3. For the second result, we identify $A_1 \otimes_{\mathbb{k}} A_1$ with $M(4, \mathbb{k})$, and the assumption on regularity allows the application of [22, Proposition 2.8], so that the line modules are in one-to-one correspondence with the elements in the span of the defining relations of A that have rank at most two. In particular, we compute $\dim(\Omega_2 \cap \Delta)$ in $\mathbb{P}(M(4, \mathbb{k}))$, where Ω_2 denotes the elements in $\mathbb{P}(M(4, \mathbb{k}))$ of rank at most two and Δ denotes the projectivization of the image in $\mathbb{P}(M(4, \mathbb{k}))$ of the span of the defining relations of A . Since $\Delta \cong \mathbb{P}^5$ and $\dim(\Omega_2) = 11$, the dimension is thus at least equal to $11 + 5 - 15 = 1$, so, generically, A has a 1-parameter family of line modules.

In spite of Van den Bergh's work, it was still not clear that a regular algebra satisfying the hypotheses from the preceding paragraph could have both a finite point scheme (especially one of cardinality twenty) and a 1-dimensional line scheme simultaneously. However, in [49], the

author proved with Van Rompay and Willaert, in the mid-1990s, that there exists a quadratic regular algebra of global dimension four on four generators with six defining relations that has exactly one point module (up to isomorphism) and a 1-parameter family of line mods.

Some years later, in 2000, Shelton and the author proved in [29] that if a quadratic algebra on four generators with six defining relations has a *finite* scheme of truncated point modules of length three, then that scheme determines the defining relations of the algebra. One should note that this result assumes no hypothesis of regularity nor of any other homological data. Moreover, by [52], this result is false in general if the scheme is infinite, even if the algebra is assumed to be regular and noetherian.

Shelton and the author also proved in [29] that if a quadratic regular algebra of global dimension four (that satisfies a few other homological conditions) has four generators and six defining relations and a 1-dimensional line scheme, then that scheme determines the defining relations of the algebra.

These last two results are counter-intuitive, since they seem to be saying that if the point scheme (respectively, line scheme) is *as small as possible*, then the defining relations can be recovered from it.

However, by the start of 2001, it was still unclear whether or not any quadratic regular algebra exists that has global dimension four, four generators, six defining relations, exactly twenty nonisomorphic point modules and a 1-dimensional line scheme. Fortunately, this was resolved by Shelton and Tingey in [28] in 2001 in the affirmative. Sadly, their method to produce their example used much trial and error on a computer, which they and others were unable to duplicate to produce more examples. This hurdle likely had a negative impact on the development of the subject, since it is difficult to make conjectures if there is only one known example. Hence, a quest began to find an algorithm to construct such algebras, but it was another several years before this situation was remedied, and that is discussed in the next section.

2. GRADED CLIFFORD ALGEBRAS, GRADED SKEW CLIFFORD ALGEBRAS AND QUANTUM PLANES

This section describes a construction of a certain type of regular algebra of arbitrary finite global dimension; such an algebra is called a graded skew Clifford algebra as it is modelled on the construction of a graded Clifford algebra. If the global dimension is four, then this construction is able to produce regular algebras that have the desired properties described at the end of the previous section. We conclude this section by revisiting the classification of quadratic regular algebras of global dimension three, and show that almost all such algebras may be obtained from regular graded skew Clifford algebras.

We continue to assume that \mathbb{k} is algebraically closed; we additionally assume $\text{char}(\mathbb{k}) \neq 2$. We write $M(n, \mathbb{k})$ for the space of $n \times n$ matrices with entries in \mathbb{k} , and M_{ij} for the entry in the $n \times n$ matrix M that is in row i and column j .

2.1. Graded Clifford Algebras.

Definition 2.1. [5, 18] Let $M_1, \dots, M_n \in M(n, \mathbb{k})$ denote symmetric matrices. A *graded Clifford algebra* (GCA) is the \mathbb{k} -algebra C on degree-one generators x_1, \dots, x_n and on degree-two generators y_1, \dots, y_n with defining relations given by:

- (i) (degree-2 relations) $x_i x_j + x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$, and
- (ii) degree-3 and degree-4 relations that guarantee y_k is central in C for all $k = 1, \dots, n$.

In general, GCAs need not be quadratic nor regular, as demonstrated by the next example.

Example 2.2. Let $M_1 = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$. The corresponding GCA is the \mathbb{k} -algebra on degree-one generators x_1, x_2 with defining relations

$$x_1 x_2 + x_2 x_1 = -x_1^2 - x_2^2, \quad x_1^2 x_2 = x_2 x_1^2,$$

so this algebra is not quadratic nor regular (as $(x_1 + x_2)^2 = 0$). For more details on this algebra, the reader may consult [47, Example 2.4].

GCAs C are noetherian by [3, Lemma 8.2], since $\dim_{\mathbb{k}}(C/\langle y_1, \dots, y_n \rangle) < \infty$. Moreover, since each matrix M_k in the definition is symmetric, we may associate a quadratic form to M_k , and thereby associate a quadric in \mathbb{P}^{n-1} to M_k for each k . This means that for each GCA, as in Definition 2.1, there is an associated quadric system \mathfrak{Q} in \mathbb{P}^{n-1} . Quadric systems are said to be base-point free if they yield a complete intersection; that is, the intersection of all the quadrics in the quadric system is empty. Although Example 2.2 demonstrates that a GCA need not be quadratic nor regular, if \mathfrak{Q} is base-point free, it determines these properties of the associated GCA as follows.

Theorem 2.3. [5, 18] *The GCA C is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the associated quadric system is base-point free; in this case, C is regular and a domain.*

In spite of this result, regular GCAs of global dimension four are not candidates for generic quadratic regular algebras of global dimension four, since, although their point schemes can be finite ([42, 49]), the symmetry of their relations prevents their line schemes from having dimension one ([29]). The standard argument to prove this for a quadratic regular GCA C of

global dimension four exploits the symmetry of the defining relations of C to move the computation of §1.4 inside $\mathbb{P}(W)$, where W is the 10-dimensional subspace of $M(4, \mathbb{k})$ consisting of all symmetric matrices. Hence, using the notation from §1.4, $\Delta \subset \mathbb{P}(W)$ and the line modules are parametrized by $(\Omega_2 \cap \mathbb{P}(W)) \cap \Delta \subset \mathbb{P}(W)$; thus the dimension is at least $6 + 5 - 9$, so it is at least two.

Hence, a modification of the definition of GCA is desired in such a way that enough symmetry is retained so as to allow an analogue of Theorem 2.3 to hold, while, at the same time, losing some symmetry so that the line scheme might have dimension one.

2.2. Graded Skew Clifford Algebras.

In order to generalize the notion of GCA and to have a result analogous to Theorem 2.3, we need to generalize the notions of symmetric matrix and quadric system and make use of normalizing sequences. For any \mathbb{N} -graded \mathbb{k} -algebra B , a sequence $\{g_1, \dots, g_m\}$ of homogeneous elements of positive degree is called *normalizing* if g_1 is a normal element in B and, for each $k = 1, \dots, m-1$, the image of g_{k+1} in $B/\langle g_1, \dots, g_k \rangle$ is a normal element.

We write \mathbb{k}^\times for $\mathbb{k} \setminus \{0\}$.

Definition 2.4. [9]

(a) Let $\mu \in M(n, \mathbb{k}^\times)$ satisfy $\mu_{ij}\mu_{ji} = 1$ for all distinct i, j . We say a matrix $M \in M(n, \mathbb{k})$ is μ -symmetric if $M_{ij} = \mu_{ij}M_{ji}$ for all $i, j = 1, \dots, n$. We write $M^\mu(n, \mathbb{k})$ for the subspace of $M(n, \mathbb{k})$ consisting of all μ -symmetric matrices.

(b) Fix μ as in (a) and additionally assume $\mu_{ii} = 1$ for all i . Let $M_1, \dots, M_n \in M^\mu(n, \mathbb{k})$. A *graded skew Clifford algebra* (GSCA) associated to μ and M_1, \dots, M_n is a graded \mathbb{k} -algebra $A = A(\mu, M_1, \dots, M_n)$ on degree-one generators x_1, \dots, x_n and on degree-two generators y_1, \dots, y_n with defining relations given by:

- (i) (degree-2 relations) $x_i x_j + \mu_{ij} x_j x_i = \sum_{k=1}^n (M_k)_{ij} y_k$ for all $i, j = 1, \dots, n$, and
- (ii) degree-3 and degree-4 relations that guarantee the existence of a normalizing sequence $\{y'_1, \dots, y'_n\}$ that spans $\sum_{k=1}^n \mathbb{k} y_k$.

Clearly, symmetric matrices and skew-symmetric matrices are μ -symmetric matrices for appropriate μ , and GCAs are GSCAs. Moreover, by [3, Lemma 8.2], GSCAs A are noetherian since $\dim_{\mathbb{k}}(A/\langle y_1, \dots, y_n \rangle) < \infty$. Furthermore, in Definition 2.4(b)(i), for all i, j , the ji -relation can be deduced from the ij -relation by the μ -symmetry of the M_k .

Examples 2.5.

(a) With μ as in Definition 2.4(b), skew polynomial rings on generators x_1, \dots, x_n with relations $x_i x_j = -\mu_{ij} x_j x_i$, for all $i \neq j$, are GSCAs.

(b) (Quantum Affine Plane) Let $n = 2$, and $M_1 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. The degree-2 relations of $A(\mu, M_1, M_2)$ have the form:

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1x_2 + \mu_{12}x_2x_1 = 0,$$

so that $\mathbb{k}\langle x_1, x_2 \rangle / \langle x_1x_2 + \mu_{12}x_2x_1 \rangle \twoheadrightarrow A(\mu, M_1, M_2)$. By Theorem 2.6 below, this map is an isomorphism (see Examples 3.2(a)).

(c) (“Jordan” Plane) Let $n = 2$, and $M_1 = \begin{bmatrix} 2 & 1 \\ \mu_{21} & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$. The degree-2 relations of $A(\mu, M_1, M_2)$ have the form:

$$2x_1^2 = 2y_1, \quad 2x_2^2 = 2y_2, \quad x_1x_2 + \mu_{12}x_2x_1 = y_1 = x_1^2,$$

so that $\mathbb{k}\langle x_1, x_2 \rangle / \langle x_1x_2 + \mu_{12}x_2x_1 - x_1^2 \rangle \twoheadrightarrow A(\mu, M_1, M_2)$. By Theorem 2.6 below, this map is an isomorphism (see Examples 3.2(b)). Depending on the choice of μ_{12} , this family of examples contains the Jordan plane and some quantum affine planes.

(d) The quadratic regular algebra of global dimension four found by Shelton and Tingey in 2001, in [28], and discussed above in §1.4, that has exactly twenty nonisomorphic point modules and a 1-dimensional line scheme is a GSCA ([9]).

One can associate a non-commutative “quadric” to each μ -symmetric matrix M_k and, in so doing, there is also a notion of “base-point free”. These ideas are discussed in §3.2 below, and yield a generalization of Theorem 2.3 as follows.

Theorem 2.6. [9] *The GSCA A is quadratic, Auslander-regular of global dimension n and satisfies the Cohen-Macaulay property with Hilbert series $1/(1-t)^n$ if and only if the associated quadric system is normalizing and base-point free; in this case, A is regular and a domain and uniquely determined, up to isomorphism, by the data μ, M_1, \dots, M_n .*

Theorem 2.6 allowed the production in [9] of many algebras that are candidates for generic quadratic regular algebras of global dimension four. In particular, there exist quadratic regular GSCAs of global dimension four on four generators with six defining relations that have exactly twenty nonisomorphic point modules and a 1-dimensional line scheme.

It is an open problem to describe the 1-dimensional line schemes of the regular GSCAs of global dimension four in [9] that have exactly twenty nonisomorphic point modules.

By Examples 1.7(c) and 2.5(b)(c), the regular algebras of global dimension at most two are GSCAs, and, by §2.3, almost all quadratic regular algebras of global dimension three are determined by GSCAs, so GSCAs promise to be very helpful in the classification of all quadratic regular algebras of global dimension four.

2.3. Quadratic Quantum Planes.

In the language of [1], a regular algebra of global dimension three that is generated by degree-1 elements is sometimes called a *quantum plane* or *quantum projective plane* or a *quantum* \mathbb{P}^2 . The classification of the *generic* quantum planes is in [2, 3, 4]. In this subsection, we summarize the results of [25], in which all *quadratic* quantum planes are classified by using GSCAs.

We continue to assume that \mathbb{k} is algebraically closed, but its characteristic is arbitrary unless specifically stated otherwise.

Let D denote a quadratic quantum plane and let $X \subset \mathbb{P}^2$ denote its point scheme. By [3, Proposition 4.3] and [25, Lemma 2.1], there are, in total, four cases to consider:

- X contains a line, or
- X is a nodal cubic curve in \mathbb{P}^2 , or
- X is a cuspidal cubic curve in \mathbb{P}^2 , or
- X is a (nonsingular) elliptic curve in \mathbb{P}^2 .

Theorem 2.7. [25] *Suppose $\text{char}(\mathbb{k}) \neq 2$. If X contains a line, then either D is a twist, by an automorphism, of a GSCA, or D is a twist, by a twisting system, of an Ore extension of a regular GSCA of global dimension two.*

Theorem 2.8. [25] *If X is a nodal cubic curve, then D is isomorphic to a \mathbb{k} -algebra on generators x_1, x_2, x_3 with defining relations:*

$$\lambda x_1 x_2 = x_2 x_1, \quad \lambda x_2 x_3 = x_3 x_2 - x_1^2, \quad \lambda x_3 x_1 = x_1 x_3 - x_2^2, \quad (*)$$

where $\lambda \in \mathbb{k}$ and $\lambda^3 \notin \{0, 1\}$. Conversely, for any such λ , any quadratic algebra with defining relations $(*)$ is a quantum plane and its point scheme is a nodal cubic curve in \mathbb{P}^2 . Moreover, if $\text{char}(\mathbb{k}) \neq 2$, then D is an Ore extension of a regular GSCA of global dimension two; in particular, if $\lambda^3 = -1$, then D is a GSCA.

Theorem 2.9. [25] *If $\text{char}(\mathbb{k}) = 3$, then X is not a cuspidal cubic curve in \mathbb{P}^2 . If $\text{char}(\mathbb{k}) \neq 3$ and if X is a cuspidal cubic curve in \mathbb{P}^2 , then D is isomorphic to a \mathbb{k} -algebra on generators x_1, x_2, x_3 with defining relations:*

$$x_1 x_2 = x_2 x_1 + x_1^2, \quad x_3 x_1 = x_1 x_3 + x_1^2 + 3x_2^2, \quad x_3 x_2 = x_2 x_3 - 3x_2^2 - 2x_1 x_3 - 2x_1 x_2. \quad (\dagger)$$

Moreover, any quadratic algebra with defining relations (\dagger) is a quantum plane; it has point scheme given by a cuspidal cubic curve in \mathbb{P}^2 if and only if $\text{char}(\mathbb{k}) \neq 3$. If $\text{char}(\mathbb{k}) \neq 2$, then any quadratic algebra with defining relations given by (\dagger) is an Ore extension of a regular GSCA of global dimension two.

It remains to discuss the case that X is an elliptic curve. In [2, 3], such algebras are classified into types A, B, E, H, where some members of each type might not have an elliptic curve as their point scheme, but a generic member does.

Theorem 2.10. [25] *Suppose that $\text{char}(\mathbb{k}) \neq 2$ and that X is an elliptic curve.*

- (a) *Quadratic quantum planes of type H are GSCAs.*
- (b) *Quadratic quantum planes of type B are GSCAs.*
- (c) *As in [2, 3], a quadratic quantum plane D of type A is given by a \mathbb{k} -algebra on generators x, y, z with defining relations:*

$$axy + byx + cz^2 = 0, \quad ayz + bzy + cx^2 = 0, \quad azx + bxz + cy^2 = 0,$$

where $a, b, c \in \mathbb{k}^\times$, $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$, $\text{char}(\mathbb{k}) \neq 3$, and either $a^3 \neq b^3$, or $a^3 \neq c^3$, or $b^3 \neq c^3$. In the case that $a^3 = b^3 \neq c^3$, D is a GSCA; whereas in the case $a^3 \neq b^3 = c^3$ (respectively, $a^3 = c^3 \neq b^3$), D is a twist, by an automorphism, of a GSCA.

In (c) of the last result, the case that $a^3 \neq b^3 \neq c^3 \neq a^3$ is still open. Moreover, the case when D is of type E is still open, but this case only consists of one algebra, up to isomorphism and anti-isomorphism. However, both type A and type E have the property that the Koszul dual of D is a quotient of a regular GSCA; so, in this sense, such algebras are weakly related to GSCAs.

3. COMPLETE INTERSECTIONS

In this section, we define the geometric terms used in Theorem 2.6. That discussion leads naturally into a consideration of a notion of non-commutative complete intersection that mimics the commutative definition.

We continue to assume that the field \mathbb{k} is algebraically closed.

3.1. Commutative Complete Intersection and Quadric Systems.

Let R denote the commutative polynomial ring on n generators of degree one. If f_1, \dots, f_m are homogeneous elements of R of positive degree, then $\{f_1, \dots, f_m\}$ is a regular sequence in R if and only if $\text{GKdim}(R/\langle f_1, \dots, f_k \rangle) = n - k \geq 0$, for all $k = 1, \dots, m$. Geometrically, this corresponds to the zero locus in $\mathbb{P}(R_1^*)$ of the ideal $J_k = \langle f_1, \dots, f_k \rangle$ having dimension $n - 1 - k \geq -1$ for all k . If $\{f_1, \dots, f_m\}$ is a regular sequence, then the zero locus of J_m (respectively, R/J_m) is called a *complete intersection* (c.f., [14]).

In the setting of §2.1, a quadric system \mathfrak{Q} is associated to symmetric matrices M_1, \dots, M_n . In that setting, \mathfrak{Q} corresponds to a regular sequence in R if and only if \mathfrak{Q} is a complete intersection, that is, if and only if \mathfrak{Q} has no base points (a base point is a point that lies on all the quadrics in \mathfrak{Q}). A non-commutative analogue of this is needed for Theorem 2.6.

3.2. Non-Commutative Complete Intersection and Quadric Systems.

The following result uses the notion of base-point module defined in Definition 1.2.

Proposition 3.1. [9, 10] *Let S denote the skew polynomial ring from §1.1, and let f_1, \dots, f_n denote homogeneous elements of S of positive degree. If $\{f_1, \dots, f_n\}$ is a normalizing sequence in S , then the following are equivalent:*

- (a) $\{f_1, \dots, f_n\}$ is a regular sequence in S ,
- (b) $\dim_{\mathbb{k}}(S/\langle f_1, \dots, f_n \rangle) < \infty$,
- (c) for each $k = 1, \dots, n$, we have $\text{GKdim}(S/\langle f_1, \dots, f_k \rangle) = n - k$,
- (d) the factor ring $S/\langle f_1, \dots, f_n \rangle$ has no right base-point modules,
- (e) the factor ring $S/\langle f_1, \dots, f_n \rangle$ has no left base-point modules.

Such a sequence $\{f_1, \dots, f_n\}$ (respectively, $S/\langle f_1, \dots, f_n \rangle$) satisfying the equivalent conditions (a)-(e) from Proposition 3.1 is called a *complete intersection* in [10].

In the setting of §2.2, one associates S to the GSCA by using μ . The isomorphism $M^\mu(n, \mathbb{k}) \rightarrow S_2$ defined by $M \mapsto (z_1, \dots, z_n)M(z_1, \dots, z_n)^T$ associates a quadric system \mathfrak{Q} to the μ -symmetric matrices M_1, \dots, M_n ; that is, \mathfrak{Q} is the span in S_2 of the images of the M_k under this map. If \mathfrak{Q} is given by a normalizing sequence in S , then it is called a *normalizing quadric system*. By Proposition 3.1, if \mathfrak{Q} is normalizing, then it corresponds to a regular sequence in S if and only if it is a complete intersection, that is, if and only if $S/\langle \mathfrak{Q} \rangle$ has no right (respectively, left) base-point modules; this is the meaning of *base-point free* in Theorem 2.6.

Examples 3.2.

(a) [9] We revisit the quantum affine plane from Examples 2.5(b), where $n = 2$. In that case, $M_i \mapsto q_i = 2z_i^2 \in S_2$, for $i = 1, 2$. The sequence $\{q_1, q_2\}$ is normalizing in S and $\dim(S/\langle q_1, q_2 \rangle) < \infty$. Thus, by Proposition 3.1, the corresponding quadric system is base-point free.

(b) [9] For Examples 2.5(c), $n = 2$ and $M_1 \mapsto q_1 = 2(z_1^2 + z_1 z_2)$ and $M_2 \mapsto q_2 = 2z_2^2$. Here, the sequence $\{q_2, q_1\}$ is normalizing in S and $\dim(S/\langle q_2, q_1 \rangle) < \infty$, so by Proposition 3.1, the corresponding quadric system is base-point free.

Proposition 3.1 has recently been extended in [47] to a family of algebras that contains the skew polynomial ring S from §1.1. In particular, an analogue of Proposition 3.1 holds for regular GSCAs, many quantum groups, and homogenizations of finite-dimensional Lie algebras.

Theorem 3.3. [47] *Let $A = \bigoplus_{i=0}^{\infty} A_i$ denote a connected, \mathbb{N} -graded \mathbb{k} -algebra that is generated by A_1 . Suppose A is Auslander-Gorenstein of finite injective dimension and satisfies the Cohen-Macaulay property, and that there exists a normalizing sequence $\{y_1, \dots, y_\nu\} \subset A \setminus \mathbb{k}$ consisting of homogeneous elements such that $\text{GKdim}(A/\langle y_1, \dots, y_\nu \rangle) = 1$. If $\text{GKdim}(A) =$*

$n \in \mathbb{N}$, and if $F = \{f_1, \dots, f_n\} \subset A \setminus \mathbb{k}^\times$ is a normalizing sequence of homogeneous elements, then the following are equivalent:

- (a) F is a regular sequence in A ,
- (b) $\dim_{\mathbb{k}}(A/\langle F \rangle) < \infty$,
- (c) for each $k = 1, \dots, n$, we have $\text{GKdim}(A/\langle f_1, \dots, f_k \rangle) = n - k$,
- (d) the factor ring $A/\langle F \rangle$ has no right base-point modules,
- (e) the factor ring $A/\langle F \rangle$ has no left base-point modules.

The reader should note that other notions of complete intersection abound in the literature, with most emphasizing a homological approach, such as the recent work in [17].

4. CONCLUSION

In this section, we list some open problems and related topics. The open problems are not listed in any particular order in regards to difficulty, and many challenge levels are included, with some quite computational in nature, and so accessible to junior researchers.

4.1. Some Open Problems.

1. As stated at the end of §2, it is still open whether or not quadratic quantum planes of type A with $a^3 \neq b^3 \neq c^3 \neq a^3$ are directly related to GSCAs; the analogous problem is also open for type E.
2. Is it possible to classify cubic quantum planes by using GSCAs, or by using an appropriate analogue of a GSCA?
3. Is it possible to classify quadratic regular algebras of global dimension four by using GSCAs? Presumably, such a classification will use both the point scheme and the line scheme.
4. Can standard results on commutative quadratic forms and quadrics be extended to non-commutative quadratic forms and quadrics? For example, P. Veerapen and the author have extended, in [50], the notion of rank of a (commutative) quadratic form to non-commutative quadratic forms on n generators, where $n = 2, 3$; can this be done for $n \geq 4$?
5. Can results concerning GCAs be carried over to GSCAs? In particular, Veerapen and the author applied their aforementioned generalization of rank to GSCAs in a way that is analogous to that used for the traditional notion of rank with GCAs in [49]. They proved in [51] that various results in [49] concerning point modules over GCAs apply to point modules over GSCAs.
6. Can standard results concerning symmetric matrices be extended or generalized to μ -symmetric matrices?
7. Can the results in [47], mentioned above at the end of §3, on complete intersections be extended to an even larger family of algebras than those considered in [47]?

8. By combining results in [9] and [42], it is known that regular GSCAs of global dimension four can have exactly N nonisomorphic point modules, where $N \notin \{2, 19\}$; it is not yet known if $N \in \{2, 19\}$ is possible. In fact, by [41], $N = 2$ is possible if the algebra is quadratic and regular of global dimension four but is not a GSCA, but it is not known if $N = 19$ is possible, even if the algebra is not a GSCA.

9. What is the line scheme of some known quadratic regular algebras of global dimension four? Such as those in [9, §5], double Ore extensions in [53, 54], generalized Laurent polynomial rings in [8], etc.

10. Does the line scheme of a generic quadratic regular algebra of global dimension four have a particular form? Perhaps a union of elliptic curves? Or, perhaps it contains at least one elliptic curve?

11. Suppose A is as in Definition 1.1 and F is as in Theorem 3.3. Let $I_k = \langle f_1, \dots, f_k \rangle$ for all $k \leq n$, and let $\widehat{\mathcal{V}(I_k)}$ denote the set of isomorphism classes of right base-point modules over A/I_k . If A is commutative, then, for each k , $\widehat{\mathcal{V}(I_k)}$ is a scheme, and so has a dimension. In particular, if A is the polynomial ring, then F is regular if and only if $\dim(\widehat{\mathcal{V}(I_k)}) = n - k - 1$, for all $k \leq n$. However, if A is not commutative, is there an analogous statement and under what hypotheses on A could it hold?

4.2. Related Topics.

Since the publication of [3], the subject has branched out in many different directions, with the key topics being: classification of regular algebras; classification of projective surfaces; seeing which commutative techniques (e.g., blowing-up, blowing-down) carry over to the non-commutative setting; and connections with differential geometry (e.g., via Poisson geometry). Module categories and homological algebra provide a unifying umbrella over these topics. These different directions are highlighted in the references cited in the Introduction and throughout the text, and in the presentations from the 2013 MSRI program found in this journal issue.

New directions continue to emerge, with one of the most recent trends being the study of regular algebras and Hopf algebras together via the consideration of Hopf actions on regular algebras, such as the work in [11]. However, perhaps the most recent exciting triumph of the subject is when the universal enveloping algebra of the Witt algebra was viewed through the geometric lens of [3] by Sierra and Walton, in [30], enabling them to solve the long-standing problem of whether or not that algebra is noetherian.

In view of all these advances, it is now clear that the marriage of non-commutative algebra and algebraic geometry, à la [3], is a dynamic and evolving field of research.

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